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Finite-dimensional representations of invariant differential operators, II ☆

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Abstract

In Part I of this paper [G.W. Schwarz, Finite-dimensional representations of invariant differential operators, J. Algebra 258 (2002) 160–204] we considered the representation theory of the algebra $\mathcal{B} := \mathcal{D}(\mathfrak{g})^G$, where $G = \mathrm{SL}_3(\mathbb{C})$ and $\mathcal{D}(\mathfrak{g})^G$ denotes the algebra of G-invariant polynomial differential operators on the Lie algebra $\mathfrak{g}$ of G. We also considered the representation theory of the subalgebra $\mathcal{A}$ of $\mathcal{B}$, where $\mathcal{A}$ is generated by the invariant functions $\mathcal{O}(\mathfrak{g})^G \subset \mathcal{B}$ and the invariant constant coefficient differential operators $S(\mathfrak{g})^G \subset \mathcal{B}$. Among other things, we found that the finite-dimensional representations of $\mathcal{A}$ and $\mathcal{B}$ are completely reducible, and we could reduce the study of the finite-dimensional irreducible representations of $\mathcal{B}$ to those of $\mathcal{A}$. Irreducible finite-dimensional representations of $\mathcal{A}$ are quotients of "Verma modules." We found sufficient conditions for the irreducible quotients of Verma modules to be finite-dimensional, and we conjectured that these sufficient conditions are also necessary. In this paper we establish the conjecture, giving a complete classification of the finite-dimensional representations of $\mathcal{A}$ and $\mathcal{B}$.

1. Introduction

Set $G := \mathrm{SL}_3(\mathbb{C})$ and $\mathfrak{g} := \mathfrak{sl}_3$, the Lie algebra of traceless 3×3 matrices. We consider the symmetric algebra $\mathrm{S}(\mathfrak{g})$ as the constant coefficient differential operators on $\mathfrak{g}$. Let e denote the quadratic invariant in $\mathcal{O}(\mathfrak{g})^G$ where $e(A) = 1/2\,\mathrm{tr}(A^2)$, $A \in \mathfrak{g}$. Let f denote minus the corresponding invariant in $\mathrm{S}(\mathfrak{g})^G$ and let E denote the Euler vector field on $\mathfrak{g}$. Then e, f and $h := [e, f] = E + 4$ are a standard basis of a copy of $\mathfrak{sl}_2$ in

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$\mathcal{B} := \mathcal{D}(\mathfrak{g})^G$, the algebra of G-invariant differential operators on $\mathfrak{g}$. We have the filtration $\mathcal{D}^n(\mathfrak{g})^G \subset \mathcal{D}^{n+1}(\mathfrak{g})^G \subset \cdots$ by order of differentiation, and the associated graded $\operatorname{gr}\mathcal{D}(\mathfrak{g})^G$ is isomorphic to $\mathcal{O}(\mathfrak{g} \oplus \mathfrak{g}^*)^G \simeq \mathcal{O}(\mathfrak{g} \oplus \mathfrak{g})^G$.

We have generators of $\mathcal{B}$ as follows. The polynomial algebra $\mathcal{O}(\mathfrak{g})^G$ is generated by e and a degree three function v_{30}, where $v_{30}(A) = \sqrt{3}/3\,\operatorname{tr}(A^3)$, $A \in \mathfrak{g}$. Applying bracket with f repeatedly, we obtain invariant differential operators $v_{21} := [f, v_{30}]$, $v_{12} := [f, v_{21}]/2$ and $v_{03} := [f, v_{12}]/3$. The span of the v_{ij} is, as $\mathfrak{sl}_2$-representation, isomorphic to the binary forms of degree 3.

Let $\mathcal{A}$ denote the subalgebra of $\mathcal{B}$ generated by $\mathcal{O}(\mathfrak{g})^G$ and $\mathrm{S}(\mathfrak{g})^G$. Clearly, e, f, h and the v_{ij} are in $\mathcal{A}$. The center of $\mathcal{B}$ is isomorphic to the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ [1,2], hence it is generated by two elements K and L. With appropriate normalizations of K and L [4, §7] we have

Theorem 1.1.

(1) $K = 1/2[v_{21}, v_{12}] - 3/2[v_{30}, v_{03}] + 4(ef + fe + h^2/2) \in \mathcal{A}$.
(2) $L^2 \in \mathcal{A}$.
(3) *$\mathcal{B}$ is a free $\mathcal{A}$-module on* 1 *and* L.
(4) *The associated graded of $\mathcal{A}$ is the polynomial algebra on* $\operatorname{gr} e$, $\operatorname{gr} f$, $\operatorname{gr} h$, $\operatorname{gr} K$ *and the* $\operatorname{gr} v_{ij}$, $i + j = 3$.

The finite-dimensional representations of $\mathcal{A}$ and $\mathcal{B}$ are completely reducible [4, 12.2, 12.4]. Since $\mathcal{B}$ is generated over $\mathcal{A}$ by 1 and L, where $L^2 \in \mathcal{A}$, the finite-dimensional irreducible representations of $\mathcal{B}$ are those of $\mathcal{A}$ with a choice of sign for L. Thus we may concentrate on determining the irreducible finite-dimensional representations of $\mathcal{A}$.

There is a PBW theorem for $\mathcal{A}$, i.e., the monomials in e, f, h, the v_{ij} and K, for any fixed ordering, form a basis of $\mathcal{A}$. We can introduce analogues of Verma modules. Let $\mathcal{A}_{<0}$ be the subalgebra of $\mathcal{A}$ generated by f, v_{12} and v_{03} and let $\mathcal{A}_{>0}$ be the subalgebra generated by e, v_{21} and v_{30}. Then a Verma module is a universal cyclic $\mathcal{A}$-module whose generator (highest weight vector) is annihilated by $\mathcal{A}_{>0}$ and is a simultaneous eigenvector for h and K. It is easy to see that any finite-dimensional irreducible $\mathcal{A}$-module has a highest weight vector, hence it is the quotient of a Verma module. Set $\bar{\kappa} = K + 7$ and $\bar{\mu} = h + 3$. (The shift makes the arithmetic simpler.) Given $\mu, \kappa \in \mathbb{C}$, let $W(\mu, \kappa)$ denote the Verma module with highest weight vector $w(\mu, \kappa)$ where $\bar{\mu} \cdot w(\mu, \kappa) = \mu w(\mu, \kappa)$ and $\bar{\kappa} \cdot w(\mu, \kappa) = \kappa w(\mu, \kappa)$. As usual, $W(\mu, \kappa)$ contains a unique maximal proper submodule $Y(\mu, \kappa)$ and corresponding simple quotient $Z(\mu, \kappa) = W(\mu, \kappa)/Y(\mu, \kappa)$ [4, 8.4].

For $(\mu, i) \in \mathbb{C}^2$, define $\kappa_1(\mu, i)$ to be $\mu^2 - \mu i + i^2$ (see Lemma 2.2). Set $V(\mu, i) := Z(\mu, \kappa_1(\mu, i))$. We say that $(\mu, i) \in \mathbb{N}^2$ is *standard* if $2 \leqslant 2i < \mu$, and *admissible* if, in addition, $\mu + i$ is not evenly divisible by 3. We showed that if $Z(\mu, \kappa)$ is finite-dimensional, then $Z(\mu, \kappa) = V(\mu, i)$ for (μ, i) standard [4, 12.1]. We showed that $V(\mu, i)$ has dimension $(1/6)\mu i(\mu - i) < \infty$ if (μ, i) is admissible [4, 12.1]. Our main theorem shows that admissibility is necessary for finite-dimensionality.

Theorem 1.2. *Let (μ, i) be standard such that* 3 *evenly divides $\mu + i$. Then* $\dim V(\mu, i) = \infty$.

Corollary 1.3.

(1) *The irreducible finite-dimensional representations of $\mathcal{A}$ are the $V(\mu,i)$ with (μ,i) admissible.*
(2) *The irreducible finite-dimensional representations of $\mathcal{B}$ are the $V(\mu,i)$, (μ,i) admissible, with a choice of sign for $L=\pm(\mu+i)(\mu-2i)(i-2\mu)$* [4, 11.2].

We established the theorem in case that μ is not evenly divisible by 3 [4, 12.1]. In this paper we give a proof that works for all μ.

Let $\mu\in\mathbb{Z}$, $\kappa\in\mathbb{C}$ and $j\in\mathbb{Z}$. Set $W(\mu,\kappa)_j=\{w\in W(\mu,\kappa)\mid\bar{\mu}\cdot w=jw\}$ and set $W(\mu,\kappa)_{[j]}=\{w\in W(\mu,\kappa)\mid h\cdot w=jw\}$. Then $W(\mu,\kappa)_j=W(\mu,\kappa)_{[j-3]}$. Define $Y(\mu,\kappa)_j$, $Y(\mu,\kappa)_{[j]}$, $Z(\mu,\kappa)_j$ and $Z(\mu,\kappa)_{[j]}$ similarly. Then $W(\mu,\kappa)=\bigoplus_j W(\mu,\kappa)_j=\bigoplus_j W(\mu,\kappa)_{[j]}$, and similarly for $Y(\mu,\kappa)$ and $Z(\mu,\kappa)$. If $i\in\mathbb{N}$ and $\kappa=\kappa_1(\mu,i)$, then we have a copy of $W(\mu-i,\kappa)$ in $W(\mu,\kappa)$ (see Theorem 2.3). Our main technical result is the following.

Proposition 1.4. *Let (μ,i) be standard but not admissible, and set $\kappa=\kappa_1(\mu,i)$. Then $Y(\mu,\kappa)_j=W(\mu-i,\kappa)_j$ for $j>-i$.*

The proposition allows us to compute the dimensions of the h-weight spaces $Z(\mu,\kappa)_{[j]}$ for $j>-i-3$. We then see that either $\dim Z(\mu,\kappa)_{[-1]}>\dim Z(\mu,\kappa)_{[1]}$ or that $\dim Z(\mu,\kappa)_{[-2]}>\dim Z(\mu,\kappa)_{[2]}$. But there would be equality if $Z(\mu,\kappa)$ were a finite-dimensional representation of $\mathfrak{sl}_2\subset\mathcal{A}$. Thus $\dim Z(\mu,\kappa)=\infty$ and we have Theorem 1.2.

In Section 2 we recall needed results from [4] and in Section 3 we give the proofs of Proposition 1.4 and Theorem 1.2.

2. Recollections

We recall some facts about Verma submodules of Verma modules $W(\mu,\kappa)$, $(\mu,\kappa)\in\mathbb{C}^2$. Verma submodules have multiplicity one [4, 8.9]. Their highest weight vectors are of the form $Rw(\mu,\kappa)$ where R is an element of $\mathcal{A}_{<0}$ and $w(\mu,\kappa)$ is a highest weight vector of $W(\mu,\kappa)$. There are three kinds of such differential operators R as we will see in Theorem 2.3 below. First we have

Proposition 2.1 [4, 8.7]. *When applied to $w(\mu,\kappa)$, L^2 acts as the scalar*

$$\lambda^2(\mu,\kappa):=\bigl(4\kappa-3\mu^2\bigr)\bigl(\kappa-3\mu^2\bigr)^2.$$

If we have a Verma submodule of $W(\mu,\kappa)$ it must be isomorphic to $W(\mu-i,\kappa)$ for some $i\in\mathbb{N}$. Then we must have

$$\begin{aligned}0&=\lambda^2(\mu,\kappa)-\lambda^2(\mu-i,\kappa)\\&=27i(-2\mu+i)\bigl(-\kappa+3\mu^2-3\mu i+i^2\bigr)\bigl(-\kappa+\mu^2-\mu i+i^2\bigr).\end{aligned}$$

We obtain

Lemma 2.2. *Suppose that there is a highest weight vector $w(\mu - i, \kappa)$ in a subquotient of $W(\mu, \kappa)$, $i > 0$. Then there are at most five possible values of i. Moreover,*

(1) $i = 2\mu$,
(2) $\kappa = \kappa_1(\mu, i) := \mu^2 - \mu i + i^2$, *or*
(3) $\kappa = \kappa_2(\mu, i) := 3\mu^2 - 3\mu i + i^2$.

A closer analysis shows that the conditions of the lemma are not sufficient for the existence of highest weight vectors in $W(\mu, \kappa)$, but they are if we insist that $i \in 2\mathbb{N}$ in (1) and that $i \in 3\mathbb{N}$ in (3). Let $\mathbb{N}^+$ denote $\mathbb{N} \setminus \{0\}$.

Theorem 2.3 [4, 9.10, 14.3, 14.5, 14.8]. *Let $(\mu, \kappa) \in \mathbb{C}^2$. Then $W(\mu, \kappa)$ is simple, unless one of the following three conditions holds.*

(1) $i \in \mathbb{N}^+$ *and* $\kappa = \kappa_1(\mu, i)$. *Then there is an element $0 \neq P_i \in \mathcal{A}_{<0}$, independent of μ, such that $P_i w(\mu, \kappa)$ is a highest weight vector $w(\mu - i, \kappa)$. The coefficient of v_{12}^i in P_i is* 1.
(2) $\mu = k \in \mathbb{N}^+$. *Then there is an element $0 \neq H_{2k} \in \mathcal{A}_{<0}$, with polynomial coefficients in κ, such that $H_{2k}(\kappa) w(\mu, \kappa)$ is a highest weight vector $w(-\mu, \kappa)$. The coefficient of v_{12}^{2k} in H_{2k} is* 1.
(3) $j \in 3\mathbb{N}^+$ *and* $\kappa = \kappa_2(\mu, j)$. *Then there is an element $0 \neq Q_j \in \mathcal{A}_{<0}$, with polynomial coefficients in μ, such that $Q_j(\mu) w(\mu, \kappa)$ is a highest weight vector $w(\mu - j, \kappa)$. The coefficient of v_{12}^j in Q_j is* 1.

The P_i, $H_{2k}(\kappa)$ and $Q_j(\mu)$ are unique with the given properties.

We computed closed expressions for the P_i and Q_j [4, 14.3, 14.5], and we calculated when one of the P_i, $Q_j(\mu)$ and $H_{2k}(\kappa)$ is a product of the others [4, 14.1]. Let $\mathcal{A}_+(\bar{\kappa} - \kappa_1(\bar{\mu}, i))$ denote the left ideal in $\mathcal{A}$ generated by e, v_{30}, v_{21} and $\bar{\kappa} - \kappa_1(\bar{\mu}, i)$. Then we have

Lemma 2.4.

(1) *Let $i \in \mathbb{N}$. Then eP_i, $v_{21}P_i$ and $v_{30}P_i$ lie in $\mathcal{A}_+(\bar{\kappa} - \kappa_1(\bar{\mu}, i))$* [4, 14.4].
(2) *Suppose that (μ, i) is standard and that* 3 *evenly divides $\mu + i$. Then $P_{\mu - i} = Q_{\mu - 2i}(\mu - i) P_i$* [4, 14.1].

Remark 2.5. Suppose that (μ, i) is standard. If $\kappa = \kappa_1(\mu, i)$, then one can compute that the highest weight vectors in any subquotient of $W(\mu, \kappa)$ can only have the $\bar{\mu}$-weights μ, $\mu - i$, i, $-i$, $-\mu + i$ and $-\mu$. Moreover, all these possibilities arise in Verma submodules of $W(\mu, \kappa)$ [4, 10.7].

The final ingredient we need is a bilinear form on $\mathcal{A}_{<0}$ and the formula for its discriminant. For this we need an involution on $\mathcal{A}$ and a projection of $\mathcal{A}$ to $\mathcal{A}_{(0)}$, the subalgebra of $\mathcal{A}$ generated by h and K. For the involution we have

Lemma 2.6 [4, 5.1]. *The action of* $\mathfrak{sl}_2$ *on* $\mathcal{A}$ *and* $\mathcal{B}$ *integrates to an action of* SL_2. *The element* $\omega := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2$ *generates the Weyl group and acts as follows*: $\omega(x_j) = \partial_j$ *and* $\omega(\partial_j) = -x_j$, $1 \leqslant j \leqslant 8$, *where* $x_1, \ldots, x_8$ *are coordinate functions on* $\mathfrak{g}$ *and* $\partial_1, \ldots, \partial_8$ *are the corresponding partial derivatives.*

Corollary 2.7 [4, 7.4]. *Let* $\sigma : \mathcal{B} \to \mathcal{B}$ *be the involutive antiautomorphism* $P \mapsto \omega(P)^t$, *where the superscript* t *denotes formal transpose and* ω *is the element of the Weyl group of* SL_2 *of Lemma* 2.6. *Then*

(1) $\sigma(e) = -f$ *and* $\sigma(f) = -e$.
(2) $\sigma(v_{ij}) = v_{ji}$, $i + j = 3$.
(3) σ *fixes* h *and* K.
(4) $\sigma(L) = -L$.

Let ${}_{-}\mathcal{A}$ denote the right ideal in $\mathcal{A}$ generated by f, v_{12} and v_{03} and let $\mathcal{A}_{+}$ denote the left ideal generated by e, v_{21} and v_{30}. Then it follows from the PBW theorem for $\mathcal{A}$ that $\mathcal{A} \simeq \mathcal{A}_{(0)} \oplus ({}_{-}\mathcal{A} + \mathcal{A}_{+})$, so we have a projection $\eta : \mathcal{A} \to \mathcal{A}_{(0)}$, and η is σ-equivariant. Using η and σ we define a bilinear form γ on $\mathcal{A}$ by

$$\gamma(\alpha, \beta) := \eta\big(\sigma(\alpha)\beta\big), \quad \alpha, \beta \in \mathcal{A}.$$

Then γ is symmetric. Define $\mathcal{A}_{<0}^{(i)}$, $i \in \mathbb{N}$, to be the weight space of weight $-i$ for the action of h on $\mathcal{A}_{<0}$. Then $\mathcal{A}_{<0}^{(i)}$ and $\mathcal{A}_{<0}^{(j)}$ are perpendicular for γ when $i \neq j$. For fixed $i \in \mathbb{N}$ let γ_i denote the restriction of γ to $\mathcal{A}_{<0}^{(i)}$ and let δ_i denote its discriminant.

Set

$$M(i) := \dim \mathcal{A}_{<0}^{(i)} \quad \text{and} \quad S(t) := \sum_{i \geqslant 0} M(i) t^i. \tag{2.7.1}$$

Note that f and v_{30} commute and that $[f, v_{12}] = 3v_{03}$ and $[v_{12}, v_{03}] = 2f^2$ [4, 7.3]. It follows that $\mathcal{A}_{<0}^{(i)}$ has basis $\{f^p v_{03}^q v_{12}^r \mid 2p + 3q + r = i\}$, hence

$$S(t) = \frac{1}{(1-t)(1-t^2)(1-t^3)}. \tag{2.7.2}$$

Theorem 2.8 [4, 9.3]. *Up to a nonzero constant,*

$$\begin{aligned} \delta_i = {} & \prod_{j=1}^{i} \big(\bar{\kappa} - \kappa_1(\bar{\mu}, j)\big)^{M(i-j)} \prod_{j=1}^{[i/2]} (\bar{\mu} - j)^{M(i-2j)} \\ & \times \prod_{j=1}^{[i/3]} \big(\bar{\kappa} - \kappa_2(\bar{\mu}, 3j)\big)^{M(i-3j)}. \end{aligned} \tag{2.8.1}$$

3. Proofs of the Main Results

Proof of Proposition 1.4. Let μ, i and κ be as in Proposition 1.4. We have a copy of $W(\mu-i,\kappa)\subset Y(\mu,\kappa)$ generated by $P_i w(\mu,\kappa)$. We want to show that $Y(\mu,\kappa)_j$ and $W(\mu-i,\kappa)_j$ are equal for $j>-i$. Consider the module $Y(\mu,\kappa)/W(\mu-i,\kappa)$. Then it follows from Remark 2.5 that $Y(\mu,\kappa)/W(\mu-i,\kappa)$ may have a highest weight vector of $\bar{\mu}$-weight i, but that if it does not, the next possible highest weight vector would have $\bar{\mu}$-weight $-i$. Thus the proposition will follow if we can show that $Y(\mu,\kappa)_i$ and $W(\mu-i,\kappa)_i$ are the same. We establish this as follows:

Choose a basis of $\mathcal{A}_{<0}^{(\mu-i)}$ where the first element is

$$P_{\mu-i}=Q_{\mu-2i}(\mu-i)P_i.$$

Set $r=M(\mu-2i)$. Choose the next $r-1$ basis elements to be of the form RP_i where $R\in\mathcal{A}_{<0}^{(\mu-2i)}$. Thus we now have a basis of $\mathcal{A}_{<0}^{(\mu-2i)}\cdot P_i$. Finally, choose $s=M(\mu-i)-M(\mu-2i)$ elements of $\mathcal{A}_{<0}^{(\mu-2i)}$ to complete a basis of $\mathcal{A}_{<0}^{(\mu-i)}$.

The matrix of $\gamma_{\mu-i}$, relative to the basis chosen above, has the form

$$\begin{pmatrix} A & B & C\\ B^t & D & E\\ C^t & E^t & F\end{pmatrix},$$

where A is 1×1, D is $(r-1)\times(r-1)$ symmetric and F is $s\times s$ symmetric (and B, etc., have the obvious dimensions). We know that any element of A, B or C is a term of the form $\gamma_{\mu-i}(x,P_{\mu-i})$, so it is divisible by $\bar{\kappa}-\kappa_1(\bar{\mu},\mu-i)$ (see Lemma 2.4). But since $P_{\mu-i}=Q_{\mu-2i}(\mu-i)P_i$, the elements of A, B and C are also divisible by $\bar{\kappa}-\kappa_1(\bar{\mu},i)$. Similarly, the entries of D and E are divisible by $\bar{\kappa}-\kappa_1(\bar{\mu},i)$. It follows that any term in the determinant of the matrix of $\gamma_{\mu-i}$ relative to our basis which is not a term of $\det A\det D\det F$ is divisible by $(\bar{\kappa}-\kappa_1(\bar{\mu},i))^a(\bar{\kappa}-\kappa_1(\bar{\mu},\mu-i))^b$ where $a\geqslant r$ and $b\geqslant 1$ and $a+b>r+1$.

We reduce to the case of one variable by fixing $\bar{\kappa}=\kappa$. Let t denote $\bar{\mu}-\mu$ and let $\tilde{\delta}_{\mu-i}$ denote $\delta_{\mu-i}$ with $\bar{\kappa}$ set to κ. Then the power of t dividing $\tilde{\delta}_{\mu-i}$ comes from the factors $\bar{\kappa}-\kappa_1(\bar{\mu},i)$ and $\bar{\kappa}-\kappa_1(\bar{\mu},\mu-i)$ of the expression (2.8.1). The other terms of $\tilde{\delta}_{\mu-i}$ give elements that do not vanish when $\bar{\kappa}=\kappa$ and $t=0$ (see [4, 10.1]). We have that $\bar{\kappa}-\kappa_1(\bar{\mu},i)$ reduces to $(-2\mu+i)t-t^2$ and $\bar{\kappa}-\kappa_1(\bar{\mu},\mu-i)$ reduces to $(-\mu-i)t-t^2$. So, $\tilde{\delta}_{\mu-i}$ is of the form $ct^{r+1}+\mathrm{O}(t^{r+2})$ where $0\neq c\in\mathbb{C}$. Any term of $\tilde{\delta}_{\mu-i}$ which does not come from $\det A\det D\det F$ is $\mathrm{O}(t^{r+2})$. Since $\det A\det D$ is of the form $ct^{r+1}+\mathrm{O}(t^{r+2})$, it follows that $c\neq 0$ and that $\det F$ is a unit in $\mathbb{C}[t]_0$, the localization of $\mathbb{C}[t]$ at $0\in\mathbb{C}$.

The techniques of a lemma of Rocha-Caridi and Wallach [3, Lemma 1] allow us to find $(r+s)\times(r+s)$ matrices R and S with values in $\mathbb{C}[t]_0$, whose determinants are nonzero constants, such that

$$R\begin{pmatrix} A & B & C\\ B^t & D & E\\ C^t & E^t & F\end{pmatrix}S=\operatorname{diag}(a_{r+s},\dots,a_s,\dots,a_1),\quad a_1,\dots,a_{r+s}\in\mathbb{C}[t]_0,$$

where the order of vanishing of $a_1, \dots, a_s$ at 0 is zero (these come from diagonalizing F) while the order of vanishing of $a_{s+1}, \dots, a_{r+s}$ at 0 increases with the index. Evaluating at $t = 0$, we see that the radical of $\gamma_{\mu-i}$ at $\bar{\kappa} = \kappa$ and $\bar{\mu} = \mu$ has dimension at most r. This radical has the same dimension as $Y(\mu,\kappa)_i$ [4, 9.1–9.2]. But we know that $W(\mu - i, \kappa)_i$ is an r-dimensional subspace of $Y(\mu,\kappa)_i$. Thus $Y(\mu,\kappa)_i = W(\mu - i, \kappa)_i$. $\square$

Proof of Theorem 1.2. Let S_k denote the Verma module for $\mathfrak{U}(\mathfrak{sl}_2)$ with highest weight $k \in \mathbb{Z}$. Now $W(\mu,\kappa)$, $\kappa = \kappa_1(\mu, i)$, has the same h-weights as $\mathcal{A}_{<0}$, only shifted up by $\mu - 3$. From the decomposition

$$\mathcal{A}_{<0}^{(i)} = \bigoplus_{2p+3q+r=i} f^p v_{03}^q v_{12}^r$$

it follows that, as a module for $\mathfrak{U}(\mathfrak{h})$ (where $\mathfrak{h} = \mathbb{C}h$), we have

$$W(\mu,\kappa) \simeq \bigoplus_{j>0} (jS_{\mu-3j} \oplus jS_{\mu-3j-1} \oplus jS_{\mu-3j-2}), \tag{3.0.2}$$

where $jS_{\mu-3j}$, etc. denotes the direct sum of j copies of $S_{\mu-3j}$, etc.

Now suppose that $\mu \equiv 0 \bmod 3$. Then $i = 3k$, $k \in \mathbb{N}^+$ and we have

$$W(\mu - i,\kappa) \simeq \bigoplus_{j>k} \bigl((j-k)S_{\mu-3j} \oplus (j-k)S_{\mu-3j-1} \oplus (j-k)S_{\mu-3j-2}\bigr), \tag{3.0.3}$$

and the coefficient of S_{-1} in (3.0.2) is k more than the coefficient of S_{-1} in (3.0.3). It follows that $\dim Z(\mu,\kappa)_{[-1]} = k + \dim Z(\mu,\kappa)_{[1]}$, which implies that $\dim Z(\mu,\kappa) = \infty$. The same argument works if $\mu \equiv 2 \bmod (3)$ and if $\mu \equiv 1 \bmod (3)$ and $i \geqslant 5$. If $\mu \equiv 1 \bmod (3)$ and $i = 2$, one shows that the coefficients of S_0 and S_{-2} are each one more in (3.0.2) than in the expansion of $W(\mu - i, \kappa)$, which implies that $\dim Z(\mu,\kappa)_{[-2]} = 2 + \dim Z(\mu,\kappa)_{[2]}$, and again we have that $\dim Z(\mu,\kappa) = \infty$. $\square$

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